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NUMERICAL INVARIANTS OF RANK-2 ARITHMETICALLY BUCHSBAUM SHEAVES

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We study rank-2 reflexive sheaves on $\mathbb{P}^3$ whose sections are Buchsbaum curves, giving vanishing theorems for the cohomology and bounds and gaps for the existence range of their Chern classes.

These results are applied to find Castelnuovo-type theorems for the cohomology of Buchsbaum curves.

1. Introduction and preliminaries

The correspondence between curves in $\mathbb{P}^3$ and rank-2 reflexive sheaves on $\mathbb{P}^3$ is well known [13, 15, 16], hence the importance of studying such sheaves. In this paper we continue the study of arithmetically Buchsbaum (briefly a.B.) sheaves, that is, reflexive sheaves whose sections are Buchsbaum curves; this work was begun in [21] and [8].

In particular, we give a vanishing theorem for unstable a.B. sheaves, which is analogous to a theorem of [21], and we apply it in order to find Castelnuovo-type theorems on Buchsbaum curves, which complete the results of the quoted paper; moreover we show that these bounds are sharp, by constructing suitable sequences of Buchsbaum curves.

Moreover, we extend part of the results of [21] to the case of a ground field of positive characteristic; in particular we show that the Frobenius pullbacks of a null-correlation bundle are not a.B.

We also study (in Section 3) the superstable case, giving vanishing theorems and bounds on the Chern classes; we also give a bound on the order of jumping lines (Section 4), which is sharp.

In the last section we study the third Chern class of an a.B. sheaf (stable or unstable), determining a lower bound and gaps in the existence range, which can be viewed as extensions of the general gaps for the third Chern class of a rank-2 reflexive sheaf on $\mathbb{P}^3$ [17].

Throughout this paper we work in $\mathbb{P}^3_k$, where k is an algebraically closed field of characteristic zero, or of characteristic $p>0$ when specified. We recall some definitions.

A coherent sheaf is said to be *reflexive* if the natural map from it to its double dual is an isomorphism [13–15]. If it has rank-2, we call it *normalized* if its first Chern class is 0 or -1.

Since the main reason for studying such sheaves (at least for rank-2 reflexive sheaves on $\mathbb{P}^3$) is the study of algebraic space curves, we are interested in sheaves with particular cohomologies.

Definitions. A rank-2 reflexive sheaf E on $\mathbb{P}^3$ is said to be *arithmetically normal* if $H^1(\mathbb{P}^3, E(t))=0$ for every t.

E is said to be *arithmetically Buchsbaum* (briefly *a.B.*) if the maps $H^1(\mathbb{P}^3, E(n)) \xrightarrow{\cdot x} H^1(\mathbb{P}^3, E(n+1))$ are zero for every n and every $x \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.

This last definition is clearly related to the definition of Buchsbaum curve. A complete reference on these topics (and more generally on the subject of Buchsbaum rings) is now available with the book [24]. Other references are [1,2,6,9]. The known results needed here are contained in [8,21,24].

A reflexive sheaf has a spectrum, that is, a sequence of integers describing its cohomology; we refer to [13,14,23] for the results needed here on this subject.

2. Vanishing theorems for a.B. sheaves

Throughout this section we work over an algebraically closed ground field of arbitrary characteristic.

Let E be a rank-2 reflexive sheaf on $\mathbb{P}^n$. We call E unstable if it fails to be stable in the sense of Mumford–Takemoto [25]. As a measure of instability of a rank-2 reflexive sheaf on $\mathbb{P}^n$ we use the following definition. A sheaf E is *unstable of order* r if r is the largest integer for which $H^0(\mathbb{P}^n, E^*(-r)) \neq 0$. Thus, unstable of order r implies $r + c_1(E) \geq 0$. See [23] for generalities on unstable reflexive sheaves.

We will give, for every unstable rank-2 a.B. sheaf E on $\mathbb{P}^3$, unstable of order r, a vanishing theorem for $H^1(\mathbb{P}^3, E(n))$ in terms of c_1, c_2 and r, thus generalizing to the unstable case of the results of [21]. Then we apply these results in order to get a Castelnuovo bound for Buchsbaum curves, which turns out to be sharp.

Lemma 2.1. *Let E be a rank-2 reflexive sheaf on $\mathbb{P}^n$, $n \geq 2$, unstable of order r. Then for a general hyperplane H in $\mathbb{P}^n$, E_H is reflexive and unstable of order r.*

Proof. See [23, Proposition 1.1]. □

From now on, we assume that E is not a direct sum of line bundles.

Lemma 2.2. *Let E be a rank-2 vector bundle on $\mathbb{P}^2$ with $c_1=0$ (resp. $c_1=-1$), unstable of order r. Then $H^1(\mathbb{P}^2, E(m))=0$ for all $m \geq c_2+r^2+r-1$ (resp. $m \geq c_2+r^2-1$).*

Proof. The proof of this lemma follows from [23, Proposition 2.1 and Remark 2.3.1]. □

Theorem 2.3. *Let E be a rank-2 a.B. sheaf on $\mathbb{P}^3$ with $c_1=0$ (resp. $c_1=-1$), unstable of order r. Then $H^1(\mathbb{P}^3, E(t))=0$ for all $t \geq c_2+r^2+r-1$ (resp. $t \geq c_2+r^2-1$).*

Proof. For every plane H in $\mathbb{P}^3$ we have an exact sequence

$$0 \to E(-1) \to E \to E_H \to 0$$

which gives us the exact cohomology sequence

$$0 \to H^0(\mathbb{P}^3, E(t-1)) \to H^0(\mathbb{P}^3, E(t)) \to H^0(H, E_H(t)) \to H^1(\mathbb{P}^3, E(t-1))$$
$$\to H^1(\mathbb{P}^3, E(t)) \to H^1(H, E_H(t)) \to H^2(\mathbb{P}^3, E(t-1)) \to \cdots.$$

By hypothesis, E is a.B., so the above exact cohomology sequence breaks up and gives us the exact sequences (as in [24], or [21] and [8])

$$0 \to H^0(\mathbb{P}^3, E(t-1)) \to H^0(\mathbb{P}^3, E(t)) \to H^0(H, E_H(t)) \to H^1(\mathbb{P}^3, E(t-1)) \to 0,$$
$$0 \to H^1(\mathbb{P}^3, E(t)) \to H^1(H, E_H(t)) \to \cdots.$$

In particular, $h^1(\mathbb{P}^3, E(t)) \leq h^1(H, E_H(t))$. Finally, using Lemmas 2.1 and 2.2, if the plane H does not pass through the singular points of E, we get $H^1(\mathbb{P}^3, E(t))=0$ for every $t \geq c_2+r^2+r-1$ (resp. $t \geq c_2+r^2-1$). □

Lemma 2.4. *Let E be a rank-2 a.B. sheaf with $c_1=0$ (resp. $c_1=-1$), unstable of order r. Then E is n-regular and $E(n)$ is generated by its global sections for every $n \geq c_2+r^2+r$ (resp. $n \geq c_2+r^2$).*

Proof. By Theorem 2.3, $H^1(\mathbb{P}^3, E(n))=0$ for every $n \geq c_2+r^2+r-1$ (resp. $n \geq c_2+r^2-1$); by Serre duality (and the order of instability) $H^3(\mathbb{P}^3, E(n))=0$ for every $n \geq r-3$; and by [23, Theorem 3.8], $H^2(\mathbb{P}^3, E(n))=0$ for every $n \geq c_2+r^2+r-2$ (resp. $n \geq c_2+r^2-2$). Hence [22, Lecture 14] for every $n \geq c_2+r^2+r$ (resp. $n \geq c_2+r^2$) E is n-regular and $E(n)$ is generated by its global sections. □

Now we are ready to apply these results to the study of Buchsbaum curves in $\mathbb{P}^3$. From now on, C will be a reduced, irreducible curve in $\mathbb{P}^3$. We denote

d = degree of C, p_a = arithmetic genus of C, $s = \min\{t \mid H^0(\mathbb{P}^3, I_C(t)) \neq 0\}$, and $e = \max\{t \mid H^1(C, \mathcal{O}_C(t) \neq 0\}$.

Theorem 2.5. *Let $C \subset \mathbb{P}^3$ be a reduced, irreducible Buchsbaum curve.*

(i) *If $e = 2m \geq 2s - 4$, then $H^1(\mathbb{P}^3, I_C(n)) = 0$ for every $n \geq d + s^2 + 2m + 3 - 2sm - 5s$.*

(ii) *If $e = 2m + 1 \geq 2s - 3$, then $H^1(\mathbb{P}^3, I_C(n)) = 0$ for every $n \geq d + s^2 + 2m + 4 - 2sm - 6s$.*

(Remark. The cases $e = 2m < 2s - 4$ and $e = 2m + 1 < 2s - 3$ were studied in [21].)

Proof. (i) We take the most negative twist of ω_C which has a nonzero section. By Serre duality on C, this is precisely $H^0(C, \omega_C(-e))$. So we take $0 \neq f \in H^0(C, \omega_C(-e))$ and we obtain a reflexive sheaf E on $\mathbb{P}^3$ as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E(m+2) \to \mathcal{I}_C(2m+4) \to 0. \tag{f}$$

The Chern classes of E can be expressed in terms of d, p_a and e by

$$c_1(E) = 0, \qquad c_2(E) = d - (m+2)^2, \qquad c_3(E) = 2p_a - 2 + ed.$$

Note that our hypothesis on e and s implies that $H^0(\mathbb{P}^3, E(s - m - 2)) \neq 0$ and $H^0(\mathbb{P}^3, E(s - m - 3)) = 0$. Therefore E is unstable of order $r = m + 2 - s$. On the other hand, a Buchsbaum curve gives rise to an a.B. sheaf. Thus applying Theorem 2.3 we get that $H^1(\mathbb{P}^3, \mathcal{I}_C(n)) = H^1(\mathbb{P}^3, E(n - m - 2)) = 0$ for every $n \geq d + s^2 + 2m + 3 - 2ms - 5s$.

(ii) We omit its proof, which is similar to the proof of (i). $\square$

Corollary 2.6. *Let $C \subset \mathbb{P}^3$ be a reduced, irreducible Buchsbaum curve.*

(i) *If $e = 2m \geq 2s - 4$, then $\mathcal{I}_C$ is n-regular and $\mathcal{I}_C(n)$ is generated by its global sections for every $n \geq \delta := \max(2m + 3, d + s^2 + 2m + 4 - 2ms - 5s)$.*

(ii) *If $e = 2m + 1 \geq 2s - 3$, then $\mathcal{I}_C$ is n-regular and $\mathcal{I}_C(n)$ is generated by its global sections for every $n \geq \delta := \max(2m + 4, d + s^2 + 2m + 5 - 2ms - 6s)$.*

Proof. The proof follows immediately from the result of [22] quoted in Lemma 2.4. $\square$

Now we recall the following:

Definition. Let $C \subset \mathbb{P}^3$ be a Buchsbaum curve. We call diameter of C, written diam(C), the number of components of the Hartshorne–Rao module of C from the first non-zero one to the last (inclusive).

It is known that if $C \subset \mathbb{P}^3$ is a Buchsbaum curve of maximal rank, then diam(C) ≤ 2 [8, 10]. So it is natural to ask if for any Buchsbaum curve $C \subset \mathbb{P}^3$ there is a bound for diam(C) in terms of the numerical invariants e, d, and s of C. In this direction, we have the following result:

Corollary 2.7. *Let $C \subset \mathbb{P}^3$ be a reduced, irreducible Buchsbaum curve.*

(i) *If* $e = 2m > 2s - 4$, *then* $\operatorname{diam}(C) \le d + s^2 + m - 2ms - 4s + 1$;

(ii) *If* $e = 2m = 2s - 4$, *then* $\operatorname{diam}(C) \le d + s^2 + m - 2ms - 5s + 1$;

(iii) *If* $e = 2m + 1 \ge 2s - 3$, *then* $\operatorname{diam}(C) \le d + s^2 + m - 2ms - 5s + 1$.

Proof. It is enough to apply Theorem 2.5 and [8, 2.7]. $\square$

Examples. In this paragraph we show that the bounds of Theorem 2.5 are sharp.

(a) For every m, we construct a Buchsbaum curve C_m such that $e = 2m = 2s - 4$ and $H^1(\mathbb{P}^3, \mathscr{I}_C(n)) \ne 0$ for $n = d + s^2 + 2m + 2 - 2sm - 5s$. More precisely, we will construct by induction on m a curve C_m such that C_m is smooth connected,

$$d(C_m) = m^2 + 4m + 6, \qquad s(C_m) = m + 2,$$

$$H^1(\mathbb{P}^3, \mathscr{I}_{C_m}(t)) \ne 0 \text{ if and only if } t = m + 2 = d + s^2 + 2m + 2 - 2sm - 5s,$$

$$e(C_m) = 2m = 2s - 4.$$

For $m = 0$, take a curve of type $(4, 2)$ on a quadric surface. Now, suppose that there exists C_m. There is a smooth surface S of degree $2m + 4$ containing C_m; consider a general plane H and a line $L \subset H$; let $P = S \cap H$. It is possible to apply the techniques of [5] and perform a liaison addition of C_m and L in such a way that $C_m \cup L \cup P$ is smoothable with fixed cohomology. It is easy to verify that the smooth curve thus obtained is C_{m+1}.

(b) For every m, we construct a smooth connected Buchsbaum curve C_m with

$$d = m^2 + 5m + 8, \qquad s = m + 2, \qquad e = 2m + 1 = 2s - 3,$$

$$H^1(\mathbb{P}^3, \mathscr{I}_C(t)) \ne 0 \quad \text{if and only if } t = m + 3 = d + s^2 + 2m + 3 - 2sm - 6s.$$

For $m = 0$, take a smooth curve of type $(5, 3)$ on a quadratic surface. In order to get C_{m+1} starting from C_m, perform a liaison addition of C_m and a line L by means of surfaces of degrees $2m + 5$ and 1 respectively, and then smooth as above.

The aim of the last paragraph of this section is to extend the results of [21] to a ground field of arbitrary characteristic, that is the vanishing results, analogous to Theorem 2.3 and Lemma 2.4, for stable a.B. sheaves. The key point is the following lemma. Now char $k = p$.

Lemma 2.8. *Let N be a null-correlation bundle, F the Frobenius and $r > 0$. Then $E = F^{r*}(N)$ is not a.B.*

Proof. We have the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to N(1) \to \mathscr{I}_{L \cup R}(2) \to 0$$

where L and R are disjoint lines. Let $q = p^r$. We may choose homogeneous coordinates x_0, x_1, x_2, x_3 such that $L=\{x_0=x_2=0\}$ and $R=\{x_1=x_3=0\}$. Let H be the plane with equation $x_2-x_3=0$, and let

$$L^{(q)}:=\{x_0^q=x_2^q=0\},$$
$$R^{(q)}:=\{x_1^q=x_3^q=0\} \quad \text{(complete intersection)},$$
$$Y^{(q)}:=L^{(q)}\cup R^{(q)}.$$

We have an exact sequence

$$0\to \mathcal{O}_{\mathbb{P}^3}\to E(q)\to \mathcal{I}_{Y^{(q)}}(2q)\to 0.$$

The homogeneous ideal of $L^{(q)}$ (resp. $R^{(q)}$) is generated by x_0^q and x_2^q (resp. x_1^q and x_3^q). We claim that $h^0(\mathbb{P}^3, \mathcal{I}_{Y^{(q)}}(q+1))=0$. Indeed, take any form $f\in H^0(\mathbb{P}^3, \mathcal{I}_{L^{(q)}}(q+1))$, $f\neq 0$, say $f=(\sum a_i x_i)x_0^q+(\sum b_i x_i)x_2^q$; then $f\notin (x_1^q, x_3^q)$ since $q>1$.

However $h^0(H, \mathcal{I}_{Y^{(q)}\cap H}(q))\neq 0$, since $(x_2+ax_3)^q\in H^0(H, \mathcal{I}_{Y^{(q)}\cap H}(q))$ for every $a\in k$. By [10, 3.2] we obtain that $Y^{(q)}$ is not a.B., and hence E is not an a.B. sheaf. □

Proposition 2.9. *Let E be a rank-2 a.B. normalized reflexive sheaf on $\mathbb{P}^3$, with Chern classes (c_1, c_2, c_3). Then $H^1(\mathbb{P}^3, E(t))=0$ for every $t\geq c_2-2-c_1$, if E is stable.*

Proof. First we assume that E is neither a nullcorrelation bundle nor a Frobenius pullback of a nullcorrelation bundle. Under this assumption, for a general plane H, E_H is stable [7, 3.2]. From the long exact sequence we get two sequences

$$0\to H^0(\mathbb{P}^3, E(t-1))\to H^0(\mathbb{P}^3, E(t))\to H^0(H, E_H(t))\to H^1(\mathbb{P}^3, E(t-1))\to 0,$$
$$0\to H^1(\mathbb{P}^3, E(t))\to H^1(H, E_H(t))\to \cdots.$$

It is enough to check that $h^1(H, E_H(t))=0$ for every $t\geq c_2-2-c_1$.

If for a general line $L\subset H$ the splitting of E_L is $\mathcal{O}_L\oplus\mathcal{O}_L(c_1)$, then the proof of [21, 1.4] works verbatim. Unfortunately, when char $k=p>0$ Grauert–Mulich's theorem does not hold. We will show that, if $E_L=\mathcal{O}_L(s)\oplus\mathcal{O}_L(c_1-s)$, with $s>0$, then $H^1(H, E_H(t))=0$ for every $t\geq c_2-s^2+c_1 s-1$.

Let $f>c_1$ be the first integer such that $h^0(H, E_H(f))\neq 0$. Hence $H^0(H, E_H(f))$ has a section, vanishing in codimension 2. If $f<s-c_1$, $E(f)_L$ has no nonvanishing section. Hence we must have $f\geq s-c_1$, that is, $h^0(H, E_H(s-c_1-1))=0$. Since by Serre's duality we have $h^2(H, E_H(s-c_1-1))=0$, by Riemann–Roch we obtain $h^1(H, E_H(s-c_1-1))=c_2-s^2-s+c_1(s+1)$.

Observe that $h^1(L, E_{L_1}(t))=0$ for every $t\geq s-c_1-1$. The same proof as [21, 1.2], shows that either $h^1(H, E_H(t))=0$ or $h^1(H, E_H(t))<h^1(H, E_H(t-1))$. Hence $h^1(H, E_H(t))=0$ for every $t\geq c_2-s^2-s+c_1(s+1)+s-c_1-1=c_2-1=c_2-s^2+sc_1-1$.

If E is a nullcorrelation bundle, then Proposition 2.9 is well known. Since the Frobenius pullbacks of a nullcorrelation bundle are not a.B. the proof of Proposition 2.9 is complete. □

Remark. As a corollary, we get that [21, 1.5, 2.2 and 2.3] are true without any restriction on char(k).

Remark. The only a.B. rank-2 vector bundles on $\mathbb{P}^3$ are direct sum of two line bundles and the null-correlation bundle.

Proof. By Ein's restriction theorem [7, Theorem 3.2] and Lemma 2.8, the proof of [8, Proposition 2.5, Lemmas 2.3, 2.4], works with no restriction on char(k).

3. The superstable case

In this chapter we give some results on superstable a.B. rank-2 sheaves on $\mathbb{P}^3$; we need no restriction on char(k).

Let E be a normalized rank-2 reflexive sheaf on $\mathbb{P}^3$; we denote $b(E):=\min\{t\in\mathbb{N}\mid H^0(\mathbb{P}^3,E(t))\neq 0\}$. So $b(E)>0$ just means that E is stable, and we call E superstable if $b(E)>1$. We want to give a bound for $b(E)$, assuming that E is a.B., and a vanishing theorem for $H^1(\mathbb{P}^3,E(t))$, as in the previous section, depending not only on c_1,c_2, but also on $b(E)$.

Lemma 3.1. *Let E be an a.B. rank-2 normalized reflexive sheaf on $\mathbb{P}^3$ and H a general plane. If $H^0(H,E_H(t))\neq 0$, then $H^0(\mathbb{P}^3,E(t+1))\neq 0$.*

Proof. Fix an integer m such that $E(m)$ has a section vanishing in codimension two, hence giving an exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^3}\to E(m)\to\mathcal{I}_Y(2m+c_1)\to 0.$$

We may take H with $\dim(Y\cap H)\leq 0$. Hence the lemma follows from [10, 3.2]. □

Proposition 3.2. *Let E be a normalized rank-2 a.B. reflexive sheaf on $\mathbb{P}^3$, with Chern classes $(0,c_2,c_3)$ (resp. $(-1,c_2,c_3)$). If $c_2\leq(h+1)(h+2)$ (resp. $c_2<(h+1)^2$), then $b(E)\leq h+1$.*

Proof. The inequalities in the hypothesis are equivalent to

$$2h^2+2hc_1-2c_2+6h+c_1^2+3c_1+4>0.$$

Hence Riemann–Roch and Serre duality in $\mathbb{P}^3$ implies that $h^0(H,E_H(h))\neq 0$, where H is a general plane. The result now follows from Lemma 3.1. □

Remark. The bound given by Proposition 3.2 for a.B. sheaves is much better than the general bound given in [13]; we recall that Hartshorne's bound is, in general, sharp (examples given in [16]).

Now observe that, given a rank-2 a.B. sheaf on $\mathbb{P}^3$, we have $h^0(\mathbb{P}^3,E(r-1))=$

$h^1(\mathbb{P}^3, E(r-2))$, with $r = b(E) - 1$, and that the maximum r for which this happens is either $b(E) - 1$, or $b(E)$ itself.

In fact, $h^0(\mathbb{P}^3, E(b(E)-2))$ is zero by definition. Suppose $h^1(\mathbb{P}^3, E(b(E)-3)) \neq 0$; this implies that for every plane $H \subset \mathbb{P}^3$ we have $h^0(H, E_H(b(E)-2)) \neq 0$, and this implies, by Lemma 3.1, $h^0(\mathbb{P}^3, E(b(E)-1)) \neq 0$, hence a contradiction.

So we study the vanishing of h^1 and the Chern classes of an a.B. sheaf such that $h^0(\mathbb{P}^3, E(r-1)) = 0 = h^1(\mathbb{P}^3, E(r-2))$, remembering that we can choose $r = b(E) - 1$ (or even maybe $r = b(E)$).

Proposition 3.3. *Let E be a normalized rank-2 a.B. reflexive sheaf on $\mathbb{P}^3$ with Chern classes (c_1, c_2, c_3), such that $h^0(\mathbb{P}^3, E(r-1)) = 0 = h^1(\mathbb{P}^3, E(r-2))$, $r > 0$. Then $h^1(\mathbb{P}^3, E(t)) = 0$ for every $t > c_2 - r^2 - rc_1 - 1$.*

Proof. Take a general plane $H \subset \mathbb{P}^3$; thanks to the usual exact sequence of restriction

$$0 \to E(-1) \to E \to E_H \to 0$$

and the Buchsbaum hypothesis, it is enough to check that for such a t we have $h^1(H, E_H(t)) = 0$. But

$$h^0(H, E_H(r-1)) = h^0(\mathbb{P}^3, E(r-1)) + h^1(\mathbb{P}^3, E(r-2)) = 0,$$

$$h^2(H, E_H(r-1)) = 0 \quad \text{(Serre duality);}$$

hence

$$h^1(H, E_H(r-1)) = -\chi(E_H(r-1)) = c_2 - r^2 - r - rc_1.$$

Thanks to [21], we have $h^1(H, E_H(s-1)) > h^1(H, E_H(s))$ if $s > 0$, hence the result. □

Looking for the Chern classes, take a normalized rank-2 reflexive sheaf E on $\mathbb{P}^3$, not necessarily a.B.

Fix a general hyperplane E; we may assume that E_H is a vector bundle. Fix an integer r and assume

$$h^0(\mathbb{P}^3, E(r-1)) = 0 = h^1(\mathbb{P}^3, E(r-2)).$$

In particular, E is stable. By the long exact sequence of cohomology we find $h^0(H, E_H(r-1)) = 0$, hence $h^0(H, E_H(t)) = 0$ for every $t < r$. By Serre duality we find $h^2(H, E_H(t)) = 0$ for every $t \geq 0$. In particular, for $0 \leq t < r$, we have

$$h^1(H, E_H(t)) = -\chi(E_H(t)) = c_2 - t^2 - 3t - 2 - c_1(t+1).$$

Let $\{k_i\}$, $i = 1, \ldots, c_2$, be the spectrum of E; let y_j, $j = 1, \ldots, z$ be the integers in $\{k_i\}$ (counted the same number of times) such that $k_i \leq -r$. Let M be a rank-z vector bundle on $\mathbb{P}^1$ with $\mathcal{O}_{\mathbb{P}^1}(y_j)$, $1 \leq j \leq z$, as factors. Thanks to the properties of the spectrum, we have

$$h^2(\mathbb{P}^3, E(t)) = h^1(\mathbb{P}^1, M(t+1)) \quad \text{for every } t \geq r-2.$$

In particular, since $H^1(\mathbb{P}^3, E(r-2))=0$, we have

$$(*) \qquad \begin{aligned} z &= h^2(\mathbb{P}^3, E(r-3)) - h^2(\mathbb{P}^3, E(r-2)) = h^1(H, E_H(r-2)) \\ &= c_2 - r^2 + r - c_1(r-1). \end{aligned}$$

Now assume that E is a.B. and $c_3>0$. It is possible to show that in this case $k_i<0$ for every i (see Lemma 5.3 below). Moreover, we have $h^0(\mathbb{P}^3, E(t)) = h^0(H, E_H(t)) = 0$ for every $t<r$, hence $h^1(\mathbb{P}^3, E(t-1))=0$ for every $t<r$. But this implies that, for every t, $0<t\leq r-1$,

$$\begin{aligned} \#\{i \mid k_i = -t\} &= h^2(\mathbb{P}^3, E(t-3)) - 2h^2(\mathbb{P}^3, E(t-2)) + h^2(\mathbb{P}^3, E(t-1)) \\ &= \chi(E(t-3)) - 2\chi(E(t-2)) + \chi(E(t-1)) = 2t + c_1. \end{aligned}$$

So, under the hypothesis $h^0(\mathbb{P}^3, E(r-1)) = 0 = h^1(\mathbb{P}^3, E(r-2))$, the spectrum giving the highest c_3 is (i):

k_i	-1	-2	$\ldots$	$-(r-1)$	$-r$	$-(r+1)$	$-(r+2)$	$\ldots$	$-(r+c_2-r^2+r-c_1(r-1)-1)$
#	$2+c_1$	$4+c_1$	$\ldots$	$2r-2+c_1$	1	1	1		1

and the spectrum giving the lowest c_3 is (ii):

k_i	-1	-2	$\ldots$	$-(r-1)$	$-r$
#	$2+c_1$	$4+c_1$	$\ldots$	$2r-2+c_1$	$c_2-r^2+r-c_1(r-1)$

These spectra correspond to the following c_3's (if $c_1=0$):

(i) $\quad c_3 = c_2^2 - (2r^2-4r+1)c_2 + \frac{1}{3}r(r-1)(3r^2-5r+1)$,

(ii) $\quad c_3 = 2rc_2 - \frac{2}{3}r(r-1)(r+1)$,

and to the following ones (if $c_1=-1$):

(i) $\quad c_3 = c_2^2 - (2r^2-6r+3)c_2 + \frac{1}{3}(r-1)(3r^3-11r^2+13r-6)$,

(ii) $\quad c_3 = (2r-1)c_2 - \frac{2}{3}r(r-1)(r-\frac{1}{2})$.

Therefore we have proved the following proposition:

Proposition 3.4. *Let E be a normalized rank-2 a.B. reflexive sheaf on $\mathbb{P}^3$ with Chern classes (c_1, c_2, c_3), such that $h^0(E, \mathbb{P}^3(r-1)) = 0 = h^1(\mathbb{P}^3, E(r-2))$, $r>1$. Then*

$$2rc_2 - \tfrac{2}{3}r(r-1)(r+1) \leq c_3 \leq c_2^2 - c_2(2r^2-4r+1) + \tfrac{1}{3}r(r-1)(3r^2-5r+1)$$

if $c_1=0$, and

$$\begin{aligned} &(2r-1)c_2 - \tfrac{2}{3}r(r-1)(r-\tfrac{1}{2}) \\ &\quad \leq c_3 \leq c_2^2 - (2r^2-6r+3)c_2 + \tfrac{1}{3}(r-1)(3r^3-11r^2+13r-6) \end{aligned}$$

if $c_1=-1$. □

4. A bound on the order of jumping lines for rank-2 a.B. sheaves on $\mathbb{P}^3$

In this section we suppose $\operatorname{char}(k)=0$.

Let E be a normalized rank-2 stable (resp. unstable of order r) reflexive sheaf on $\mathbb{P}^3$. By Grothendieck's theorem, the restriction of E to a line $L\subset\mathbb{P}^3$ is isomorphic to $\mathcal{O}_L(b_L+c_1)\oplus\mathcal{O}_L(-b_L)$, for some $b_L\geq 0$. This b_L is an upper semicontinuous function of L which takes the value 0 (resp. r) on an open set of $G(1,3)$ [11] (resp. [23]). If $b_L>0$ (resp. $b_L>r$), then L is said to be a jumping line of order b_L. The goal of this section is to give a sharp bound on b_L, for normalized rank-2 stable (resp. unstable of order r) a.B. sheaves on $\mathbb{P}^3$, in terms of $c_1(E)$ and $c_2(E)$ (resp. $c_1(E)$, $c_2(E)$, r).

Lemma 4.1. *Let E be a rank-2 stable reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ (resp. $c_1=-1$). If E has a jumping line L of order $b>c_2-1$ (resp. $b>c_2$), then $H^1(\mathbb{P}^3,E(b-2))\neq 0$.*

Proof. Let $H\supset L$ be any plane. We consider the exact sequence

$$0\to E_H(-1)\to E_H\to E_L\to 0$$

which gives the exact cohomology sequence

$$\cdots\to H^1(H,E_H(b-2))\to H^1(L,E_L(b-2))\to H^2(H,E_H(b-3))\to\cdots.$$

The last term vanishes. In fact, if $H^2(H,E_H(b-3))\neq 0$, then H is an unstable plane for E of order t, $t\geq b\geq c_2$, which contradicts [20, Proposition 2]. Hence $H^1(L,E_L(b-2))\neq 0$ implies $H^1(H,E_H(b-2))\neq 0$.

On the other hand, the exact cohomology sequence of

$$0\to E(-1)\to E\to E_H\to 0$$

gives that $H^1(\mathbb{P}^3,E(b-2))\to H^1(H,E_H(b-2))$ is surjective, since $H^2(\mathbb{P}^3,E(b-2))=0$ [13, 8.2]. Therefore $H^1(\mathbb{P}^3,E(b-2))\neq 0$. □

Lemma 4.2. *Let E be a rank-2 reflexive sheaf on $\mathbb{P}^3$ with $c_1=0$ (resp. $c_1=-1$), unstable of order r. If E has a jumping line L of order $b>c_2+r^2+r$ (resp. $b>c_2+r^2$), then $H^1(\mathbb{P}^3,E(b-2))\neq 0$.*

Proof. The proof is exactly as in Lemma 4.1; we only have to consider [20, Proposition 5] and [23, Theorem 3.8] instead of [20, Proposition 2] and [13, Theorem 8.2]. □

Proposition 4.3. *Let E be a normalized rank-2 stable a.B. sheaf on $\mathbb{P}^3$. Then the order of any jumping line is $\leq c_2-c_1-1$.*

Proof. Let b denote the order of a jumping line L. If $b>c_2-c_1-1$, then $H^1(\mathbb{P}^3,E(b-2))\neq 0$ (Lemma 4.1), which contradicts [21, Theorem 1.4]. □

Examples. Here and after Proposition 4.4 we construct sheaves as extensions of arithmetically normal curves. Hence $H^1(\mathbb{P}^3, E(t))=0$ for every t (and hence E is a.B.).

(a) For every $c_2 \geq 2$, we will give an example of a rank-2 stable a.B. sheaf E on $\mathbb{P}^3$ with $c_1=0$, $c_2(E)=c_2$, having a jumping line of order c_2-1, thus showing that the result is sharp.

To this end, we construct a rank-2 reflexive sheaf E on $\mathbb{P}^3$ as an extension

$$\text{(e)} \qquad 0 \to \mathcal{O}_{\mathbb{P}^3} \to E(1) \to \mathcal{I}_Y(2) \to 0$$

where Y is the union of a plane curve Y_1 of degree c_2 with a line D meeting Y_1 at one point, and $0 \neq e \in H^0(Y, \omega_Y(2))$ is a global section which generates $\omega_Y(2)$ except at finitely many points. It is not difficult to see that E is a rank-2 stable a.B. sheaf on $\mathbb{P}^3$ with Chern classes $(0, c_2, c_2^2-c_2+2)$. Let L be a line such that $h^0(Y \cap L, \mathcal{O}_{Y \cap L})=c_2$. Tensoring with $\mathcal{O}_L(-1)$ the above exact sequence, we get a surjective map

$$E_L = \mathcal{O}_L(b_L) \oplus \mathcal{O}_L(-b_L) \to \mathcal{O}_L(1-c_2) \to 0.$$

Since $\operatorname{Hom}(\mathcal{O}_L(b_L), \mathcal{O}_L(1-c_2))=0$, the $\mathcal{O}_L(-b_L)$ summand maps to $\mathcal{O}_L(1-c_2)$ surjectively, hence $b_L=c_2-1$.

(b) For every c_2, we will give an example of a rank-2 stable a.B. sheaf E on $\mathbb{P}^3$ with $c_1(E)=-1$, $c_2(E)=c_2$, having a jumping line of order c_2. To this end, we construct a rank-2 reflexive sheaf E on $\mathbb{P}^3$ as an extension

$$\text{(e)} \qquad 0 \to \mathcal{O}_{\mathbb{P}^3} \to E(1) \to \mathcal{I}_Y(1) \to 0$$

where Y is a smooth plane curve of degree c_2 and $0 \neq e \in H^0(Y, \omega_Y(3))$ is a global section which generates $\omega_Y(3)$ except at finitely many points.

It is not difficult to see that E is a rank-2 stable a.B. sheaf on $\mathbb{P}^3$ with Chern classes $(-1, c_2, c_2^2)$.

Let $L \subset \mathbb{P}^3$ be a line meeting Y at c_2 points. Tensoring the above exact sequence by $\mathcal{O}_L(-1)$, we get a surjective map

$$E_L = \mathcal{O}_L(b_L-1) \oplus \mathcal{O}_L(-b_L) \to \mathcal{O}_L(-c_2) \to 0.$$

Since $\operatorname{Hom}(\mathcal{O}_L(b_L-1), \mathcal{O}_L(-c_2))=0$, the $\mathcal{O}_L(-b_L)$ summand maps to $\mathcal{O}_L(-c_2)$ surjectively, hence $b_L=c_2$.

Proposition 4.4. *Let E be a rank-2 a.B. sheaf on $\mathbb{P}^3$ with $c_1=0$ (resp. $c_1=-1$), unstable of order r. Then the order of any jumping line is $\leq c_2+r^2+r$ (resp. $\leq c_2+r^2$).*

Proof. Let b denote the order of a jumping line. If $b > c_2+r^2+r$ (resp. $b>c_2+r^2$), then $H^1(\mathbb{P}^3, E(b-2)) \neq 0$ (Lemma 4.2), which contradicts Theorem 2.3. □

Examples. (a) For all c_2, r, such that $r \geq 0$, $c_2+r^2>0$ we wish to show that there is a rank-2 a.B. sheaf E on $\mathbb{P}^3$ with $c_1(E)=0$, $c_2(E)=c_2$, unstable of order r, having a jumping line of order c_2+r^2+r. To this end, we construct a rank-2 reflexive sheaf E on $\mathbb{P}^3$ as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E(-r) \to \mathcal{I}_Y(-2r) \to 0 \tag{e}$$

where Y is a smooth plane curve of degree c_2+r^2 and $0 \neq e \in H^0(Y, \omega_Y(4+2r))$ is a global section which generates $\omega_Y(4+2r)$ except at finitely many points. It is not difficult to see that E is a rank-2 a.B. sheaf on $\mathbb{P}^3$, unstable of order r, with Chern classes $(0, c_2, (c_2+(r+1)^2)(c_2+r^2))$. The same argument as before shows that any line L meeting Y at c_2+r^2 points is a jumping line of order c_2+r^2+r.

(b) For all c_2, r such that $r \geq 1$, $c_2+r^2-r>0$, we wish to construct a rank-2 a.B. sheaf E on $\mathbb{P}^3$ with $c_1(E)=-1$, $c_2(E)=c_2$, unstable of order r, having a jumping line of order c_2+r^2. To this end, we construct as usual a rank-2 reflexive sheaf E on $\mathbb{P}^3$ as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E(-r) \to \mathcal{I}_Y(-2r-1) \to 0 \tag{e}$$

where Y is a plane curve of degree c_2+r^2-r and $0 \neq e \in H^0(Y, \omega_Y(5+2r))$ is a global section which generates $\omega_Y(5+2r)$ except at finitely many points. As above, it is easy to show that E is a rank-2 a.B. sheaf on $\mathbb{P}^3$, unstable of order r, with Chern classes $(-1, c_2, (c_2+r(r-1))(c_2+r(r+1)))$. The same argument as above shows that any line L meeting Y at c_2+r^2-r points is a jumping line of order c_2+r^2.

Hence these examples show that the bounds given in this section are sharp.

5. Third Chern class of an arithmetically Buchsbaum sheaf

A natural problem in studying reflexive sheaves is to determine the Chern classes of these sheaves. A general answer was given, for rank-2 and rank-3 reflexive sheaves on $\mathbb{P}^3$, by one of the authors [13, 17, 19]. Here we want to study the same problem for rank-2 arithmetically Buchsbaum sheaves on $\mathbb{P}^3$, showing that the Buchsbaum condition implies a further restriction on c_3, given c_1 and c_2.

Throughout this section we assume $\operatorname{char}(k)=0$.

Consider for instance the case $c_1=0$. Hartshorne [13] showed that $0 \leq c_3 \leq c_2^2-c_2+2$, and in the papers quoted above it was shown that there are gaps below the upper bound. We will see that in our situation (a) there is a better lower bound for c_3, (b) new gaps appear, (c) the old gaps become larger.

(a) There is a better lower bound for c_3

Proposition 5.1. *Let F be a rank-2 arithmetically Buchsbaum sheaf on $\mathbb{P}^3$ with Chern classes $(0, c_2, c_3)$, $(t+1)(t+2) \leq c_2 < (t+2)(t+3)$, $t \geq 1$. Then $c_3 \geq f(c_2)$, where $f(c_2)$ is an increasing function of c_2.*

Proof. For every s, $t+1 \leq s \leq c_2 - 1$, let

$$A(s, c_2) = 2(s+2)c_2 - \tfrac{2}{3}(s+1)(s+2)(s+3),$$

$$B(s, c_2) = 2c_2 + s^2 - s,$$

$$X(s, c_2) = \max\{A(s, c_2), B(s, c_2)\},$$

and denote

$$f(c_2) = \min\{X(s, c_2) \mid t+1 \leq s \leq c_2 - 1\}.$$

Notice that, for fixed c_2, $B(s, c_2)$ is a strictly increasing function of s, and that $A(s, c_2)$ is a strictly decreasing function of s, for fixed c_2, in the interval $[t+1, c_2 - 1]$. In fact, since

$$(t+1)(t+2) \leq c_2 < (t+2)(t+3),$$

the function

$$\chi(F(s)) = \tfrac{1}{2}c_3 - (s+2)c_2 + \tfrac{1}{3}(s+1)(s+2)(s+3)$$

is strictly increasing for $s \geq t+1$ (see [4]), and therefore $A(s, c_2)$ is strictly decreasing for $s \geq t+1$.

Moreover, if $c_2 > 6$ we have $A(t+1, c_2) \geq B(t+1, c_2)$ (and $>$ holds if $c_2 > 7$), and $A(c_2 - 1, c_2) < B(c_2 - 1, c_2)$.

Now we can suppose that $\chi(F(t)) < 0$. In fact, if $\chi(F(t)) \geq 0$, then

$$c_3 \geq 2(t+2)c_2 - \tfrac{2}{3}(t+1)(t+2)(t+3),$$

and this is bigger than or equal to $A(t+1)$, and so in that case the theorem is proved.

But now $\chi(F(t)) < 0$ implies $h^1(\mathbb{P}^3, F(t)) \neq 0$, and this implies [8] that $h^2(\mathbb{P}^3, F(t-2)) \neq 0$. Since

$$h^2(\mathbb{P}^3, F(t-2)) = \bigoplus h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k_i + t - 1)),$$

where $\{k_i\} = \operatorname{Spec} F$, we know that in $\operatorname{Spec} F$ there is at least one element less or equal to $-(t+1)$.

Denote by $-s$ the minimum integer appearing in $\operatorname{Spec} F$. By the previous discussion, $(t+1) \leq s$, and in general s must be less or equal to $c_2 - 1$ [13].

Now we have two conditions on $c_3(F)$.

First of all, the minimal spectrum possible is

$$(-s, -(s-1), -(s-2), \ldots, -2, \underbrace{-1, -1, \ldots, -1}_{c_2 - s \text{ times}})$$

which gives $c_3 = 2[(c_2 - s) + \tfrac{1}{2}s(s-1)] = B(s, c_2)$, and therefore $c_3(F) \geq B(s, c_2)$.

On the other hand, we have $h^2(\mathbb{P}^3, F(s-2)) = 0$, and this implies $h^1(\mathbb{P}^3, F(s)) = 0$ [8]. Therefore

$$\chi(F(s))=h^0(\mathbb{P}^3,F(s))=\tfrac{1}{2}c_3-(s+2)c_2-\tfrac{1}{3}(s+1)(s+2)(s+3)\geq 0.$$

Hence $c_3(F)\geq A(s,c_2)$.

So we have proved that $c_3(F)\geq X(s,c_2)$.

Since s varies in the interval $[t+1,c_2-1]$, the first part of the proposition is proved.

In order to prove that $f(c_2)$ is strictly increasing, it is enough to observe that A, B, and therefore X, are increasing functions, for fixed s, of the secund argument. So, if c_2 and c_2+1 are relative to the same t, we obtain $f(c_2)<f(c_2+1)$.

If $c_2=(t+2)(t+3)-1$, it is enough to check that $f(c_2)$ is not $X(t+1,c_2)$. But

$$\begin{aligned}X(t+1,c_2)=A(t+1,c_2)&=2(t+3)(t^2+5t+5)-\tfrac{2}{3}(t+2)(t+3)(t+4)\\&>B(t+2,c_2).\end{aligned}$$

Hence $X(t+1,c_2)>X(t+2,c_2)$. This completes the proof. □

Proposition 5.2. *Let F be a rank two arithmetically Buchsbaum sheaf on $\mathbb{P}^3$ with Chern classes $(-1,c_2,c_3)$, $(t+1)^2\leq c_2<(t+2)^2$, $t\geq 1$. Then $c_3\geq f'(c_2)$, where $f'(c_2)$ is an increasing function of c_2, defined as*

$$f'(c_2)=\min\{X'(s,c_2)\mid t+1\leq s\leq c_2-1\},$$

where

$$X'(s,c_2)=\max\{A'(s,c_2),B'(s,c_2)\},$$

$$A'(s,c_2)=(2s+3)c_2-\tfrac{1}{3}(s+1)(s+2)(2s+3),$$

$$B'(s,c_2)=c_2+s^2-s.$$

Proof. The same as above. □

Remark. This bound is far from being optimal, but it is general and quite easy to compute (in the appendix we give a simple Pascal program for it). Better bounds could be found by using the notion of Buchsbaum invariant of an a.B. sheaf, derived from the notion of Buchsbaum invariant of a Buchsbaum curve.

(b) and (c) New gaps appear and the old gaps become larger

We recall here that it is known [17] that $(c_1,c_2,c_3)\in\mathbb{Z}^3$ are the Chern classes of a normalized rank-2 stable reflexive sheaf E on $\mathbb{P}^3$ if and only if

(i) $c_1\in\{0,-1\}$;

(ii) $c_2\geq 0$;

(iii) $(c_1,c_2,c_3)\neq(0,1,2)$;

(iv) $c_1c_2\equiv c_3 \pmod 2$;

(v) $0\leq c_3\leq c_2^2$ if $c_1=-1$, $0\leq c_3\leq c_2^2-c_2+2$ if $c_1=0$;

(vi) If $c_1=-1$, then

$$c_3\notin\bigcup_{r=1}^{b(-1,c_2)}\,]c_2^2-2rc_2+2r(r+1),\,c_2^2-2(r-1)c_2[,$$

if $c_1=0$, then

$$c_3 \notin \bigcup_{r=1}^{b(0,c_2)}]c_2^2-(2r+1)c_2+2(r+1)^2, c_2^2-(2r-1)c_2)[,$$

where $b(0,c_2)=[-1+(c_2-2)^{1/2}]$ and $b(-1,c_2)=[(-1+(4c_2-7)^{1/2})/2]$.

Now, we will find how the gaps (vi) become larger and new gaps appear.

Lemma 5.3. *Let E be a normalized rank-2 stable a.B. sheaf on $\mathbb{P}^3$ with Chern classes (c_1,c_2,c_3) and spectrum $\{k_i\}$, $i=1,\dots,c_2$. Then $k_i\le -1$ for every i.*

Proof. By [8, 2.3], $H^1(\mathbb{P}^3,E(t))=0$ for every $t\le -1$. On the other hand, the definition of the spectrum says that $H^1(\mathbb{P}^3,E(-1))=0$ means $k_1\le -1$ for every $i=1,\dots,c_2$. □

Notation.

$$A(-1,c_2)=[\tfrac{1}{2}(-3+(8c_2-5)^{1/2})], \qquad A(0,c_2)=[\tfrac{1}{2}(-3+(8c_2-15)^{1/2})].$$

Proposition 5.4. (a) *For every $c_2\ge 4$, $1\le r\le A(-1,c_2)$, there is no rank-2 stable a.B. sheaf E on $\mathbb{P}^3$ with Chern classes $(-1,c_2,c_3)$ satisfying*

$$c_2^2-2rc_2+2r(r+1)<c_3<c_2{}^2-2(r-1)c_2+r^2-r$$

(b) *For every $c_2\ge 5$, $1\le r\le A(0,c_2)$, there is no rank-2 stable a.B. sheaf E on $\mathbb{P}^3$ with Chern classes $(0,c_2,c_3)$ satisfying*

$$c_2^2-(2r+1)c_2+2(r+1)^2<c_3<c_2^2-(2r-1)c_2+r^2+r.$$

Remark. The hypotheses on c_2 and r in the statement are to guarantee that the conditions

$$c_2^2-2rc_2+2r(r+1)<c_3<c_2^2-2(r-1)c_2+r^2-r$$

and

$$c_2^2-(2r+1)c_2+2(r+1)^2<c_3<c_2^2-(2r-1)c_2+r^2+r$$

are non-empty.

Proof of Proposition 5.4. (a) Under the hypotheses of Proposition 5.4, $c_2^2-2rc_2+2r(r+1)>\frac{1}{2}c_2^2+c_2$. So we may assume that we have a rank-2 stable a.B. sheaf E on $\mathbb{P}^3$ with Chern classes $(-1,c_2,c_3)$ and $c_3>\frac{1}{2}c_2^2+c_2$. Let $\{k_i\}$ be the spectrum of E. By [14, 5.2], E has an unstable plane H of order c_2-r for some r, $0\le r\le\frac{1}{2}(c_2-3)$, and $k_1=-(c_2-r)$. On the other hand, using Proposition 4.3 and the formula $c_3=-2\sum_i k_i-c_2$, we get that the minimum and the maximum possible values of c_3 for given r correspond to the spectra

$$-(c_2-r), -(c_2-r)+1,\dots,-2,\underbrace{-1,-1,\dots,-1}_{r+1\text{ times}}$$

and

$$-(c_2-r), -(c_2-r)+1, \ldots, -r-2, -r-1, -r-1, -r, -r, \ldots, -2, -2, -1,$$

which give $c_3 = c_2^2 - 2rc_2 + r^2 + r$ and $c_3 = c_2^2 - 2rc_2 + 2r(r+1)$ respectively.

This implies that there are no rank-2 stable a.B. sheaves on $\mathbb{P}^3$ with Chern classes $(-1, c_2, c_3)$ satisfying

$$c_2^2 - 2rc_2 + 2r(r+1) < c_3 < c_2^2 - 2(r-1)c_2 + r^2 - r.$$

Notice that for this condition to be non-empty it is necessary that $c_2 \geq 4$ and $1 \leq r \leq A(-1, c_2)$.

(b) We omit the proof, which is similar to the proof of (a). □

Remark. Conditions (i)–(vi), together with the restriction imposed by Proposition 5.4, do not suffice to assure the existence of a rank-2 stable a.B. sheaf E on $\mathbb{P}^3$ with given Chern classes, as we saw in Proposition 5.1.

Notice moreover that the gaps pointed out in Proposition 5.4 for small r have the same lower bound as the general gaps, and a higher upper bound. From $r = b(c_1, c_2) + 1$ to $A(c_1, c_2)$ the gaps are new.

Examples. Consider for instance $c_1 = 0$, $c_2 = 8$. We have $b(0,8) = 1$, $A(0,8) = 2$. Therefore there is a general gap which says that there are no stable reflexive sheaves with $c_1 = 0$, $c_2 = 8$ and $c_3 \in]48, 56[$. This gap increases, since there is no a.B. sheaf with $c_3 = 56$ and there is a new gap, corresponding to $r = 2$, that is, there is no a.B. sheaf with $c_3 = 44$.

It is easy to see that there actually exist arithmetically normal (hence a.B.) sheaves with $c_3 = 48$ and $c_3 = 42$. The first one is constructed by means of an arithmetically normal curve Y with numerical character $(12, 11, 10, 9, 8, 8, 7)$ and a section of $\omega_Y(-8)$; the second one by means of an arithmetically normal curve X with numerical character $(10, 9, 8, 8, 7, 6)$ and a section of $\omega_X(-6)$.

Notice that for $c_3 = 46$ there is no arithmetically normal sheaf; this follows immediately from the classification of arithmetically normal curves given in [12].

It is possible to show that there is only one (cohomologically speaking) a.B. sheaf with Chern classes $(0, 8, 46)$. In fact, the only possible spectrum is $(-1, -1, -1, -2, -3, -4, -5, -6)$, which gives the following cohomology:

	-2	-1	0	1	2	3	4
h^0	0	0	0				
h^1	0	0	1				
h^2	23	15	10	6	3	1	0
χ	23	15	9	7	11	23	

This implies that $h^0(\mathbb{P}^3, F(1)) \geq 1$, $h^0(\mathbb{P}^3, F(2)) \geq 8$. Consider a good section of $F(1)$, with the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to F \to \mathcal{I}_Y(1) \to 0,$$

where Y is a Buchsbaum curve of degree 9 such that $h^0(\mathbb{P}^3, \mathcal{I}_Y(3)) = 4$. This implies [1,2] that the Buchsbaum invariant of Y, that is the sum of all dimensions $h^1(\mathbb{P}^3, \mathcal{I}_Y(n))$, is 1. Therefore $h^1(\mathbb{P}^3, \mathcal{I}_Y(1)) = 1$, $h^1(\mathbb{P}^3, \mathcal{I}_Y(s)) = 0$ for every $s \neq 1$, and the complete cohomology of F must be

	-2	-1	0	1	2	3	4
h^0	0	0	0	1	8	22	$h^0(F(4))$
h^1	0	0	1	0	0	0	0
h^2	23	15	10	6	3	1	0
χ	23	15	9	7	11	23	$h^0(F(4))$

A sheaf with this cohomology can be constructed by means of a maximal rank curve $X \in L_1$ (see [3]) with numerical character $(12, 11, 10, 9, 8, 8, 7)$ and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-6) \to F \to \mathcal{I}_X(6) \to 0.$$

Compare this with the case $(0, 8, 48)$: these two curves have plane sections with the same postulation, but different genera (hence the difference on c_3).

Of course, the same problem of finding the Chern classes can be studied also in the unstable case. Conditions (necessary and sufficient) can be found, in the general case, in [18]. As before, we will find a further restriction to be imposed on the Chern classes of a normalized rank-2 a.B. sheaf E on $\mathbb{P}^3$, unstable of order r, involving also the order of unstability. All sheaves are assumed to be indecomposable.

Lemma 5.5. *Let E be a normalized rank-2 a.B. sheaf on $\mathbb{P}^3$, unstable of order r, with Chern classes (c_1, c_2, c_3), spectrum $\{k_i\}$, $i = 1, \ldots, c_2 + c_1 r + r^2$. Then $k_i \leq -1 - r$ for every i.*

Proof. By [8, 2.4], we have $H^1(\mathbb{P}^3, E(t)) = 0$ for every $t \leq r - 1$. On the other hand [23, 3.1], $0 = h^1(\mathbb{P}^3, E(r-1)) = h^0(\mathbb{P}^1, \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(k_i + r))$. So $k_i \leq -1 - r$ for every $i = 1, \ldots, c_2 + rc_1 + r^2$. $\square$

Corollary 5.6. *Let E be a normalized rank-2 a.B. sheaf on $\mathbb{P}^3$, unstable of order r. Then*

(i) $c_3 \geq 2(1 + r)(c_2 + r^2)$ *if* $c_1 = 0$,

(ii) $c_3 \geq (2r + 1)(c_2 + r^2 - r)$ *if* $c_1 = -1$.

Proof. Follows from Lemma 3.5 and [23, 3.6]. $\square$

Let now

$$A(-1, c_2, r) = [\tfrac{1}{2}(-3 + (-7 + 8(c_2 + r^2 - r))^{1/2})],$$

$$A(0, c_2, r) = [\tfrac{1}{2}(-3 + (-7 + 8(c_2 + r^2))^{1/2})].$$

Proposition 5.7. (i) *For all* c_2, r, t *such that* $r \geq 1$, $c_2 + r^2 - r \geq 4$, $1 \leq t \leq A(-1, c_2, r)$, *there are no rank-2 a.B. sheaves on* $\mathbb{P}^3$, *unstable of order* r, *with Chern classes* $(-1, c_2, c_3)$ *satisfying*

$$c_2^2 + r^4 + 2c_2r^2 - r^2 - 2t(c_2 + r^2 - r - t - 1) < c_3$$
$$< c_2^2 + r^4 + 2c_2r^2 - r^2 - 2(t-1)(c_2 + r^2 - r) + t^2 - t.$$

(ii) *For all* c_2, r, t *such that* $r \geq 0$, $c_2 + r^2 \geq 4$, $1 \leq t \leq A(0, c_2, r)$ *there are no rank-2 a.B. sheaves on* $\mathbb{P}^3$, *unstable of order* r, *with Chern classes* $(0, c_2, c_3)$ *satisfying*

$$c_2^2 + r^4 + 2c_2r^2 + 2rc_2 + 2r^3 + c_2 + r^2 - 2t(c_2 + r^2 - t - 1) < c_3$$
$$< c_2^2 + r^4 + 2c_2r^2 + 2rc_2 + 2r^3 + c_2 + r^2 - 2(t-1)(c_2 + r^2 - 1) + (t-1)(t-2).$$

Proof. We omit the proof, which is similar to the proof of Proposition 5.4. □

Remark. These gaps also appear as a generalization of the gaps given in [18].

Appendix

Program for finding a lower bound for c_3 ($c_1 = 0$) (see Proposition 5.1).

```
program Buchsbaum0 (input, output);
  var
    c, t, x, s, a, b, f: integer;
begin
  writeln('c1 = 0, c2 = ?');
  read(c);
  writeln;
  t := 1;
  repeat
    x := t * t + 3 * t + 2;
    t := t + 1;
  until x > c;
  s := t - 1;
  repeat
    a := 2 * (s + 2) * c - 2 * (s + 1) * (s + 2) * (s + 3) div 3;
    b := 2 * c + s * s - s;
    s := s + 1;
  until b > a;
  if 2 * s * c - 2 * (s - 1) * s * (s + 1) div 3 > 2 * c + (s - 1) * (s - 1) - (s - 1)
  then
    f := 2 * s * c - 2 * (s - 1) * s * (s + 1) div 3
```

```
  else
    f := 2 * c + (s - 1) * (s - 1) - (s - 1);
  writeln('c1 = 0, c2 = ', c);
  writeln('f(c2) = ', f);
end.
```

References

[1] M. Amasaki, On the structure of arithmetically Buchsbaum curves in $\mathbb{P}_k^3$, Publ. Res. Inst. Math. Sci. 20 (1984) 793–837.

[2] M. Amasaki, Integral arithmetically Buchsbaum curves in $\mathbb{P}^3$, Preprint, Res. Inst. Math. Sci. Kyoto 1987.

[3] G. Bolondi, On the classification of curves linked to two skew lines, in: Proc. Conference on Algebraic Geometry, Berlin 1985, Teubner-Text zur Mathematik 92 (Teubner, Leipzig, 1986) 38–52.

[4] G. Bolondi, Arithmetically normal sheaves, Bull. Soc. Math. France 135 (1) (1987) 71–95.

[5] G. Bolondi, Smoothing curves by curvilinear sheaves, Proc. Amer. Math. Soc. 102 (4) (1988) 797–803.

[6] G. Bolondi and J.C. Migliore, Classification of maximal rank curves in the liaison class L_n, Math. Ann. 277 (1987) 585–603.

[7] L. Ein, Stable vector bundles on projective spaces in char $p > 0$, Math. Ann. 254 (1980) 53–72.

[8] Ph. Ellia and M. Fiorentini, Courbes arithmetiquement Buchsbaum de l'espace projectif, Ann. Univ. Ferrara Sez. VII (N.S.) 33 (1988) 89–111.

[9] A. Geramita, P. Maroscia and W. Vogel, A note on arithmetically Buchsbaum curves in $\mathbb{P}^3$, Queen's University Preprint No. 1983-24.

[10] A. Geramita and J.C. Migliore, On the ideal of an arithmetically Buchsbaum curve, J. Pure Appl. Algebra 54 (1988) 215–247.

[11] H. Grauert and G. Mülich, Vektorbündel vom Rank 2 über dem n-dimensionalen Komplex projectiven Raum, Manuscripta Math. 16 (1975) 75–100.

[12] L. Gruson and Ch. Peskine, Genre des courbes de l'espace projectif, in: Algebraic Geometry Proceedings (Tromsø, 1977), Lecture Notes in Mathematics 687 (Springer, Berlin, 1978) 31–59.

[13] R. Hartshorne, Stable reflexive sheaves, Math. Ann. 254 (1980) 121–176.

[14] R. Hartshorne, Stable reflexive sheaves II, Invent. Math. 66 (1982) 165–190.

[15] R. Hartshorne, On the classification of algebraic space curves, II, Preprint.

[16] R. Hartshorne and A. Hirschowitz, Nouvelles courbes de bon genre dans l'espace projectif, Prepublicationes Mathématiques Université de Nice 116 (1986).

[17] R.M. Miró-Roig, Gaps in Chern classes of rank-2 stable reflexive sheaves, Math. Ann. 270 (1985) 317–323.

[18] R.M. Miró-Roig, Chern classes of rank-2 reflexive sheaves, J. London Math. Soc. 33 (1986) 421–429.

[19] R.M. Miró-Roig, Chern classes of rank-3 stable reflexive sheaves, Math. Ann. 276 (1987) 291–302.

[20] R.M. Miró-Roig, Unstable planes for rank-2 reflexive sheaves, Proc. Roy. Soc. Edinburgh 105 A (1987) 161–166.

[21] R.M. Miró-Roig, On the theorem of Castelnuovo for Buchsbaum curves, Preprint, Barcelona, 1987.

[22] D. Mumford, Lectures on curves on an algebraic surface, Annals of Mathematics Studies 59 (1986).

[23] T. Sauer, Nonstable reflexive sheaves on $\mathbb{P}^3$, Trans. Amer. Math. Soc. 281 (1984) 633–655.

[24] J. Stuckrad and W. Vogel, Buchsbaum Rings and Applications (VEB Deutscher Verlag der Wissenschaften, Berlin and Springer, Berlin, 1986).

[25] F. Takemoto, Stable vector bundles on algebraic surfaces, Nagoya Math. J. 47 (1972) 29–48.